

The Smaller (SALI) and the Generalized (GALI) Alignment Index Methods of Chaos Detection: Theory and Applications

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Outline

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 - ✓ **Behavior for chaotic and regular motion**
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Definition of Smaller Alignment Index (SALI)

Consider the **n-dimensional phase space of a conservative dynamical system** (**symplectic map or Hamiltonian flow**).

An orbit in that space with initial condition :

$$P(0)=(x_1(0), x_2(0), \dots, x_n(0))$$

and a **deviation vector**

$$v(0)=(dx_1(0), dx_2(0), \dots, dx_n(0))$$

The evolution in time (in maps the time is discrete and is equal to the number **N** of the iterations) of a **deviation vector** is defined by:

- the **variational equations** (for Hamiltonian flows) and
- the equations of the **tangent map** (for mappings)

Definition of SALI

We follow the evolution in time of two different initial deviation vectors ($\mathbf{v}_1(0)$, $\mathbf{v}_2(0)$), and define SALI (Skokos, 2001, J. Phys. A, 34, 10029) as:

$$\text{SALI}(t) = \min \left\{ \left\| \hat{\mathbf{v}}_1(t) + \hat{\mathbf{v}}_2(t) \right\|, \left\| \hat{\mathbf{v}}_1(t) - \hat{\mathbf{v}}_2(t) \right\| \right\}$$

where

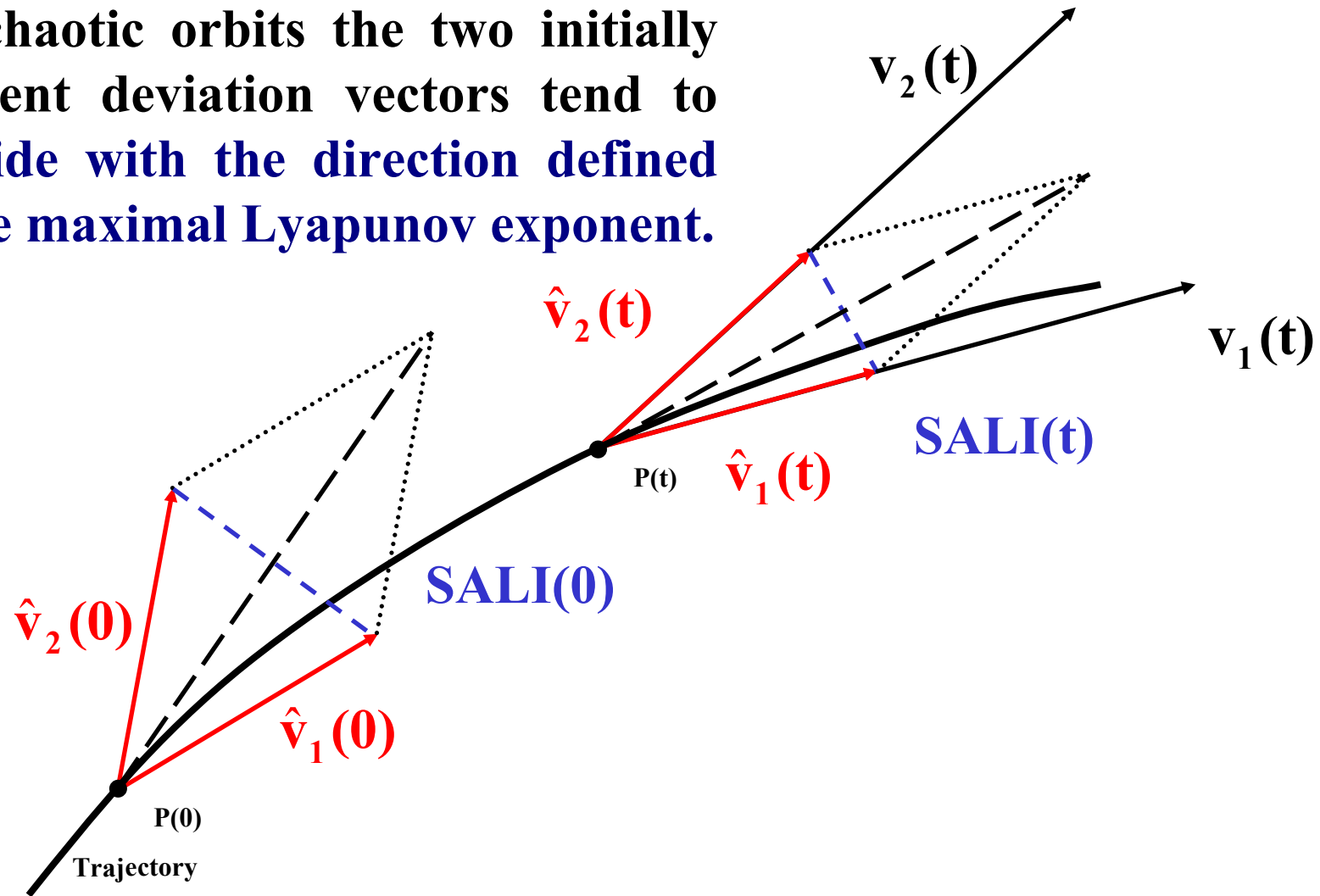
$$\hat{\mathbf{v}}_1(t) = \frac{\mathbf{v}_1(t)}{\left\| \mathbf{v}_1(t) \right\|}$$

When the two vectors become **collinear**

$$\text{SALI}(t) \rightarrow 0$$

Behavior of SALI for chaotic motion

For chaotic orbits the two initially different deviation vectors tend to coincide with the direction defined by the maximal Lyapunov exponent.

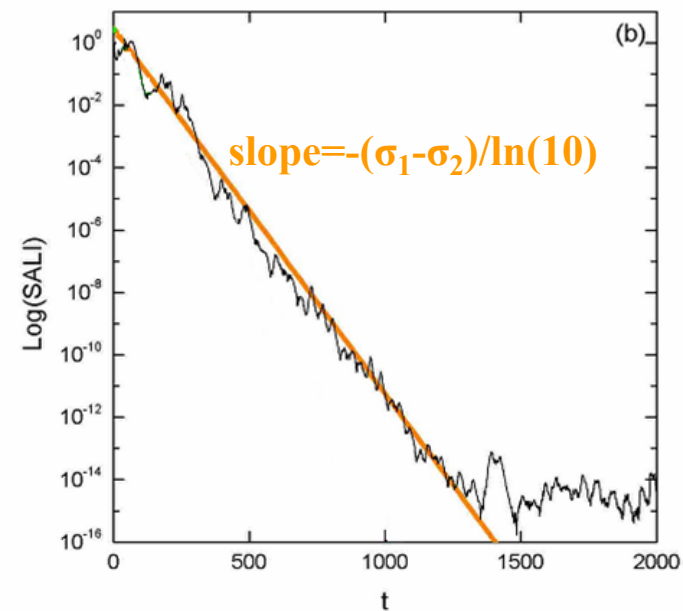
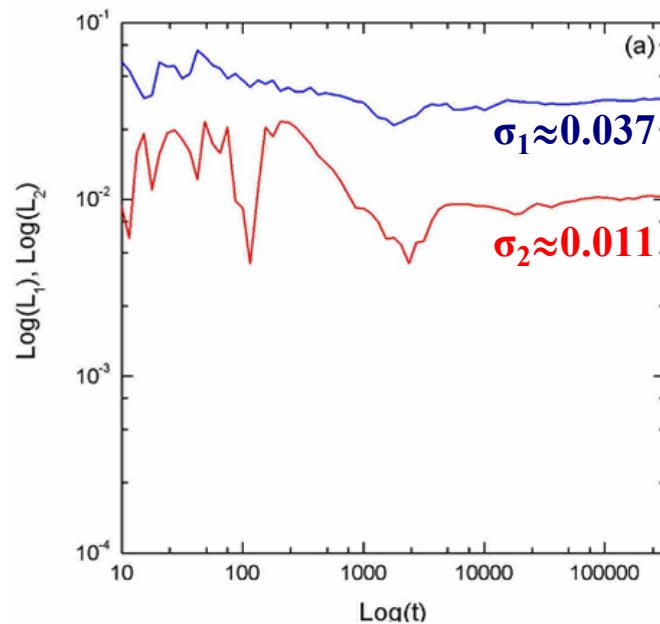


Behavior of SALI for chaotic motion

We test the validity of the approximation $\text{SALI} \propto e^{-(\sigma_1 - \sigma_2)t}$ (Skokos et al., 2004, J. Phys. A, 37, 6269) for a chaotic orbit of the 3D Hamiltonian

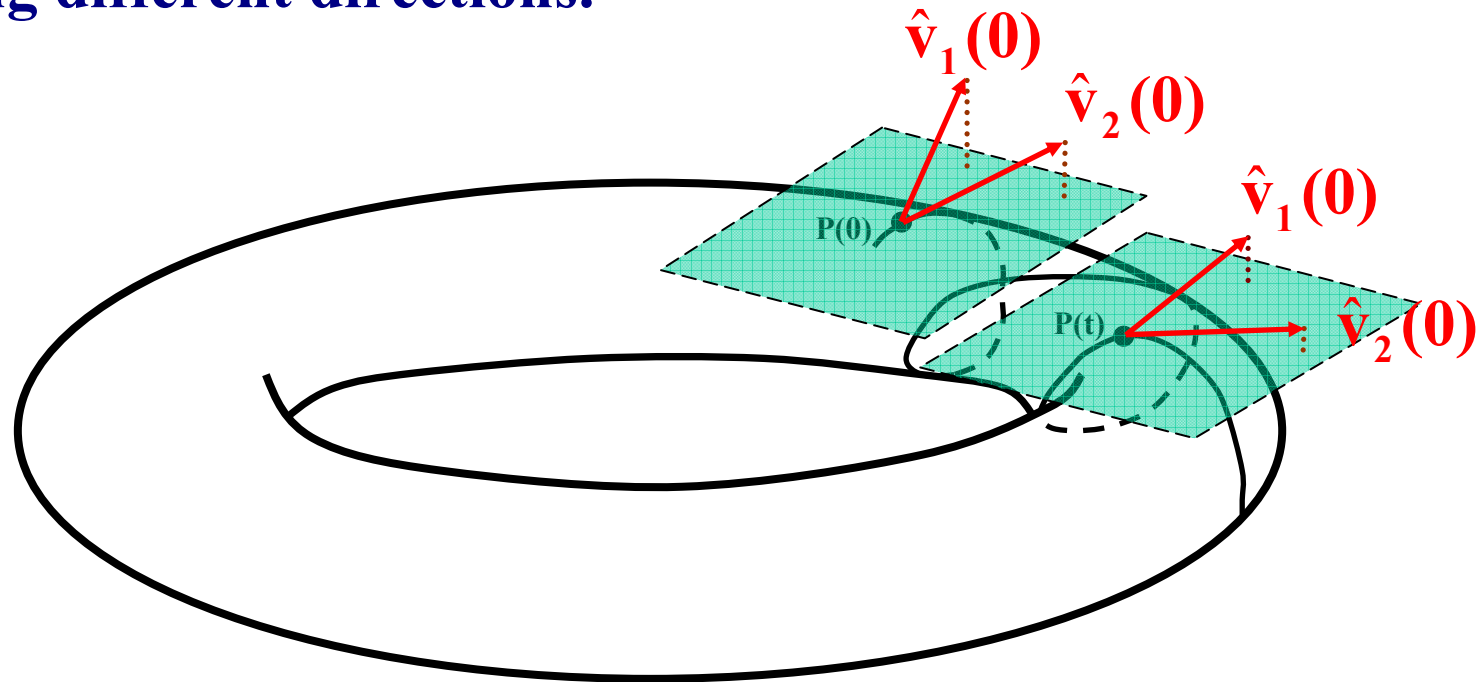
$$H = \sum_{i=1}^3 \frac{\omega_i}{2} (q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

with $\omega_1=1$, $\omega_2=1.4142$, $\omega_3=1.7321$, $H=0.09$



Behavior of SALI for **regular motion**

Regular motion occurs on a torus and two different initial deviation vectors **become tangent to the torus, generally having different directions.**



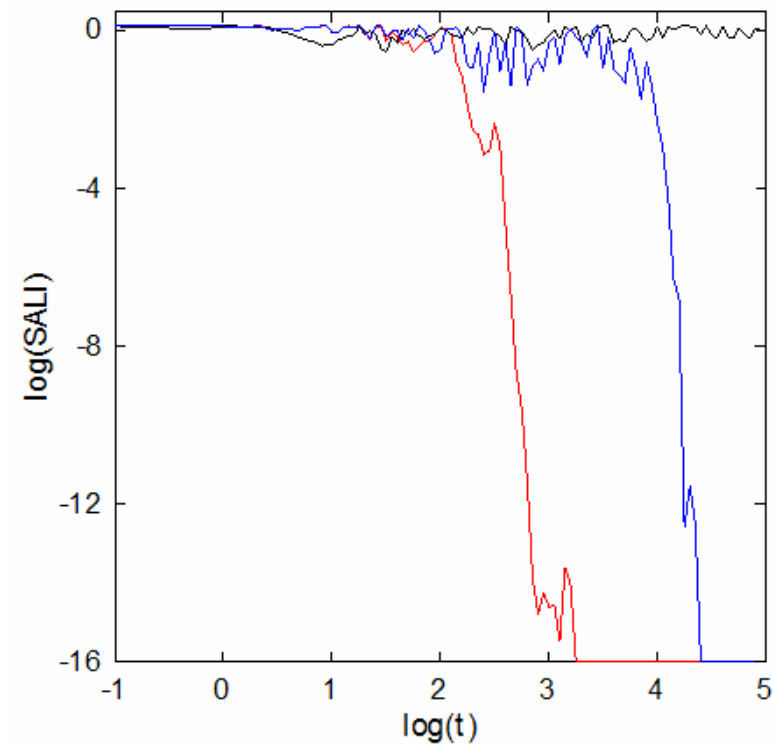
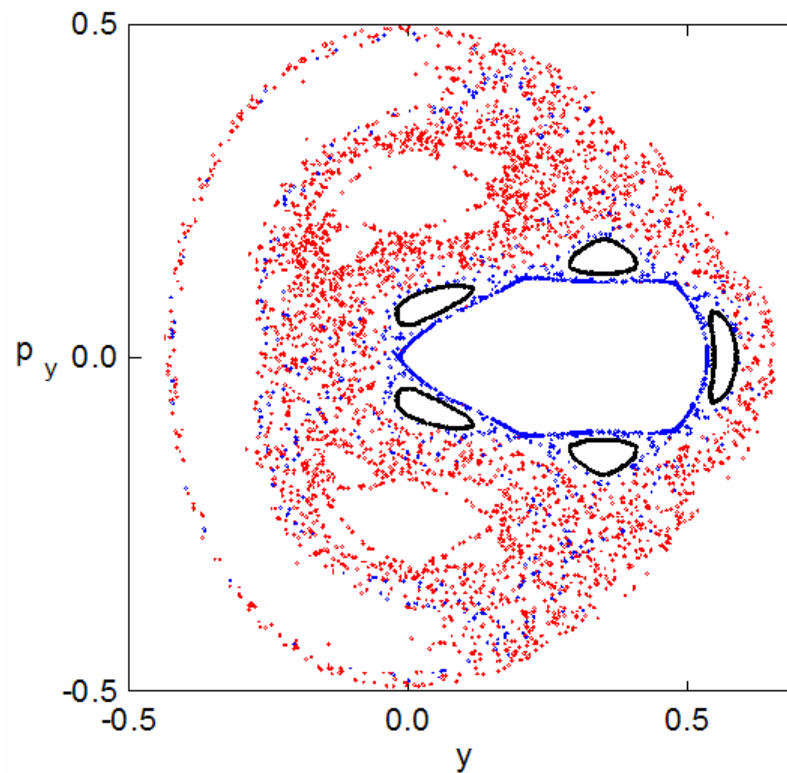
Applications – Hénon-Heiles system

For $E=1/8$ we consider the orbits with initial conditions:

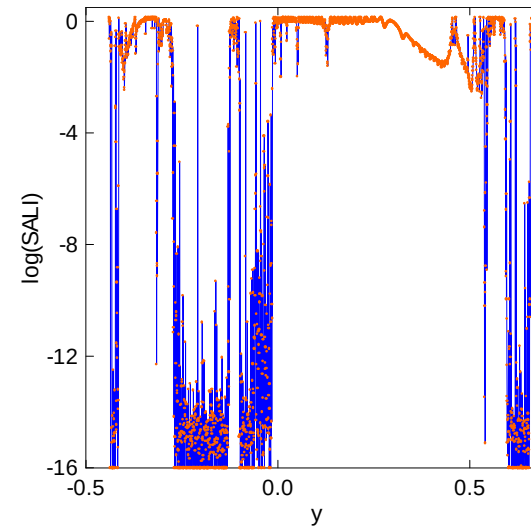
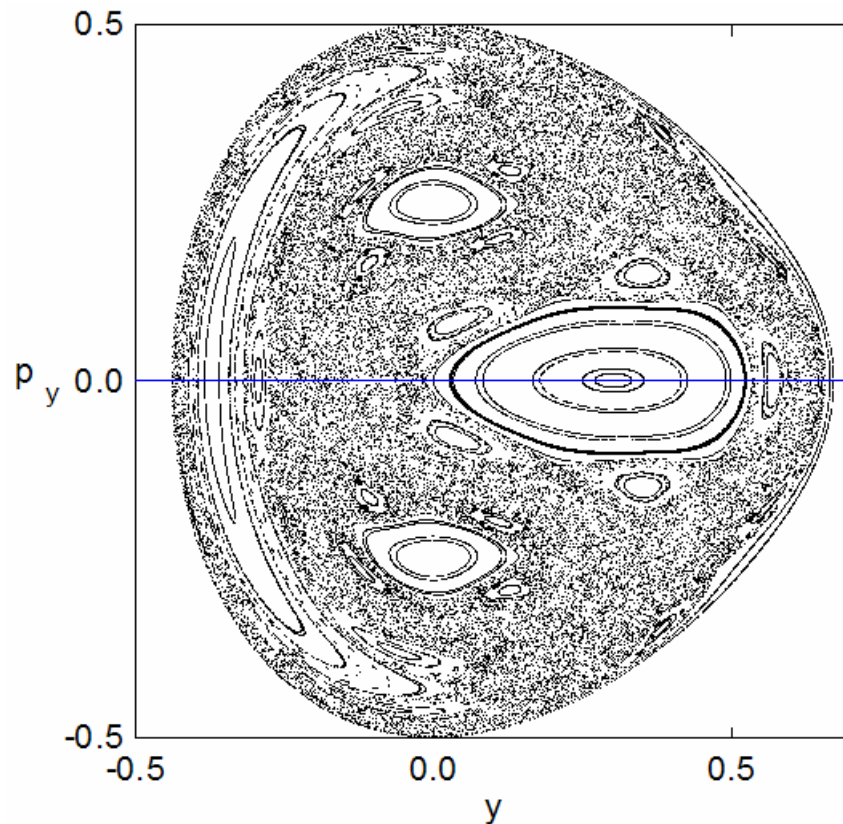
Ordered orbit, $x=0$, $y=0.55$, $p_x=0.2417$, $p_y=0$

Chaotic orbit, $x=0$, $y=-0.016$, $p_x=0.49974$, $p_y=0$

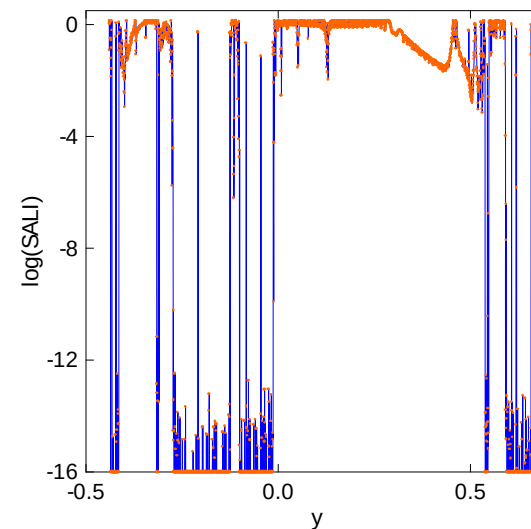
Chaotic orbit, $x=0$, $y=-0.01344$, $p_x=0.49982$, $p_y=0$



Applications – Hénon-Heiles system

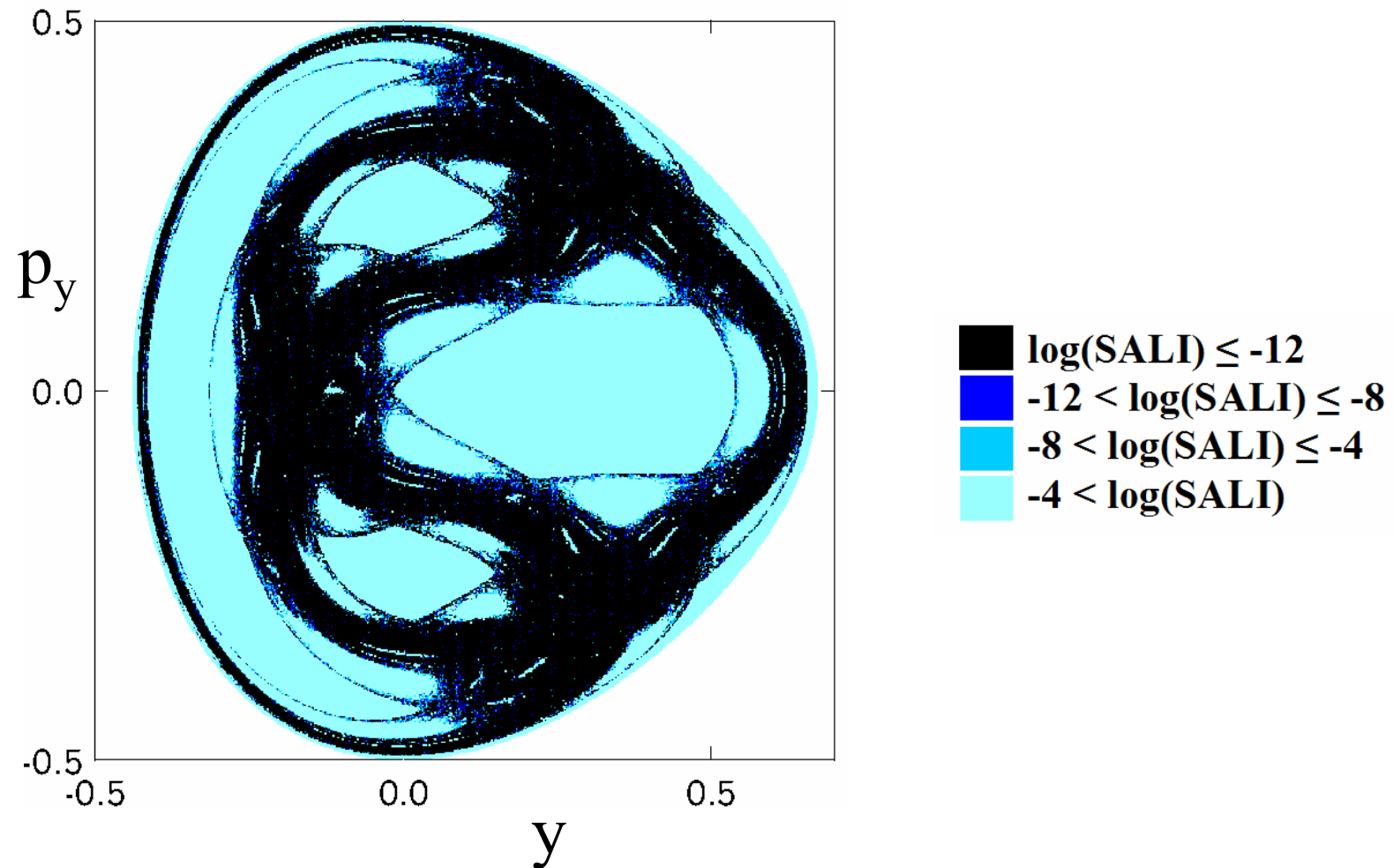


$t=1000$



$t=4000$

Applications – Hénon-Heiles system



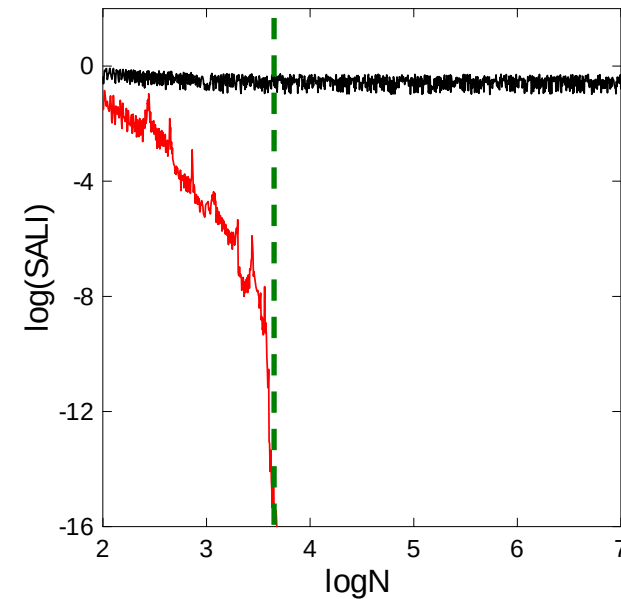
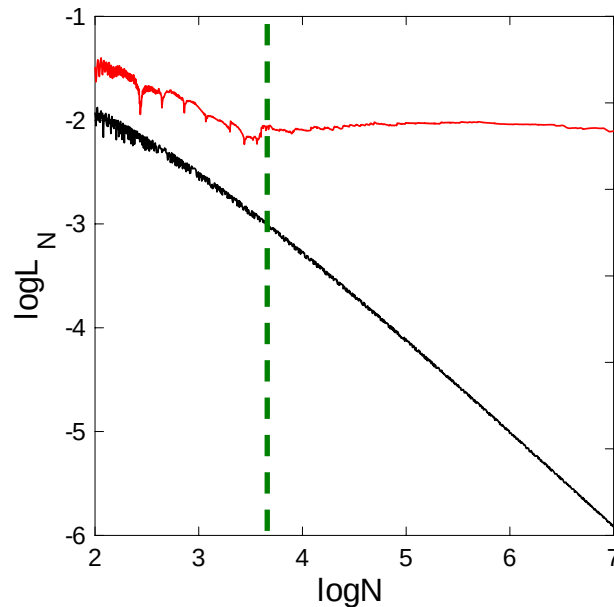
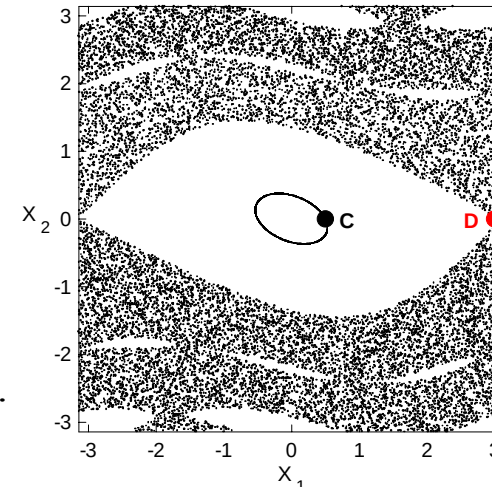
Applications – 4D map

$$\begin{aligned}
 x_1' &= x_1 + x_2 \\
 x_2' &= x_2 - \nu \sin(x_1 + x_2) - \mu [1 - \cos(x_1 + x_2 + x_3 + x_4)] \\
 x_3' &= x_3 + x_4 \\
 x_4' &= x_4 - \kappa \sin(x_3 + x_4) - \mu [1 - \cos(x_1 + x_2 + x_3 + x_4)]
 \end{aligned} \pmod{2\pi}$$

For $\nu=0.5$, $\kappa=0.1$, $\mu=0.1$ we consider the orbits:

ordered orbit C with initial conditions $x_1=0.5$, $x_2=0$, $x_3=0.5$, $x_4=0$.

chaotic orbit D with initial conditions $x_1=3$, $x_2=0$, $x_3=0.5$, $x_4=0$.



Applications – 4D Accelerator map

We consider the 4D symplectic map

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} \cos\omega_1 & -\sin\omega_1 & 0 & 0 \\ \sin\omega_1 & \cos\omega_1 & 0 & 0 \\ 0 & 0 & \cos\omega_2 & -\sin\omega_2 \\ 0 & 0 & \sin\omega_2 & \cos\omega_2 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 + x_1^2 - x_3^2 \\ x_3 \\ x_4 - 2x_1x_3 \end{pmatrix}$$

describing the **instantaneous sextupole ‘kicks’** experienced by a **particle** as it **passes through an accelerator** (Turchetti & Scandale 1991, Bountis & Tompaidis 1991, Vrahatis et al. 1996, 1997).

x_1 and x_3 are the particle’s deflections from the ideal circular orbit, in the horizontal and vertical directions respectively.

x_2 and x_4 are the associated momenta

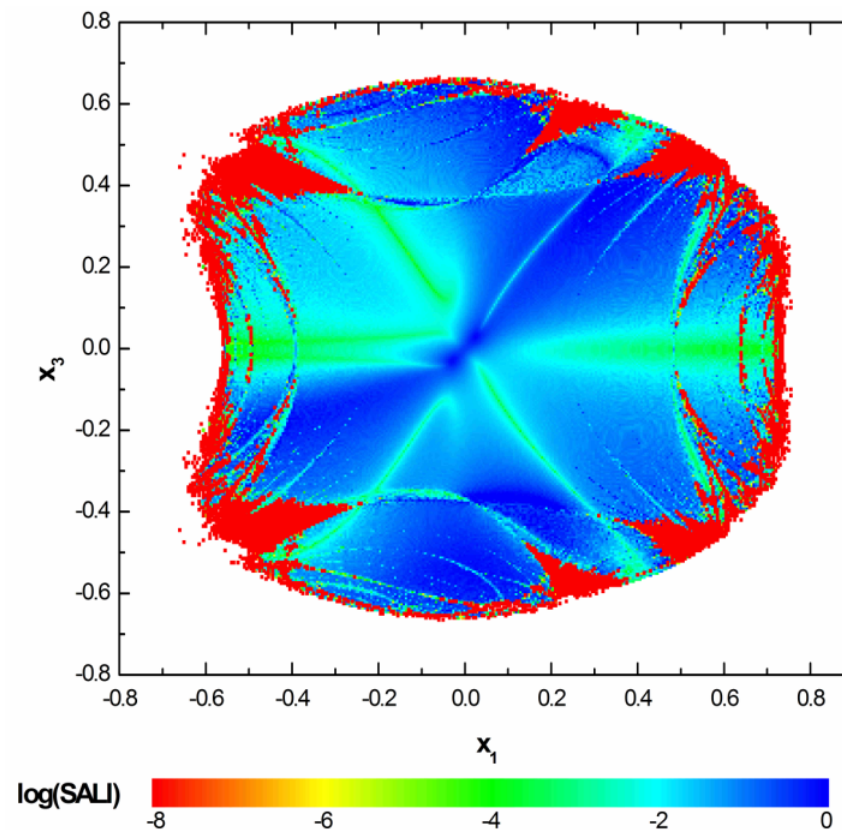
ω_1, ω_2 are related to the accelerator’s tunes q_x, q_y by

$$\omega_1 = 2\pi q_x, \quad \omega_2 = 2\pi q_y$$

Our problem is to estimate the **region of stability** of the particle’s motion, the so-called **dynamic aperture** of the beam (Bountis & Skokos, 2006, Nucl. Inst Meth. Phys Res. A, 561, 173).

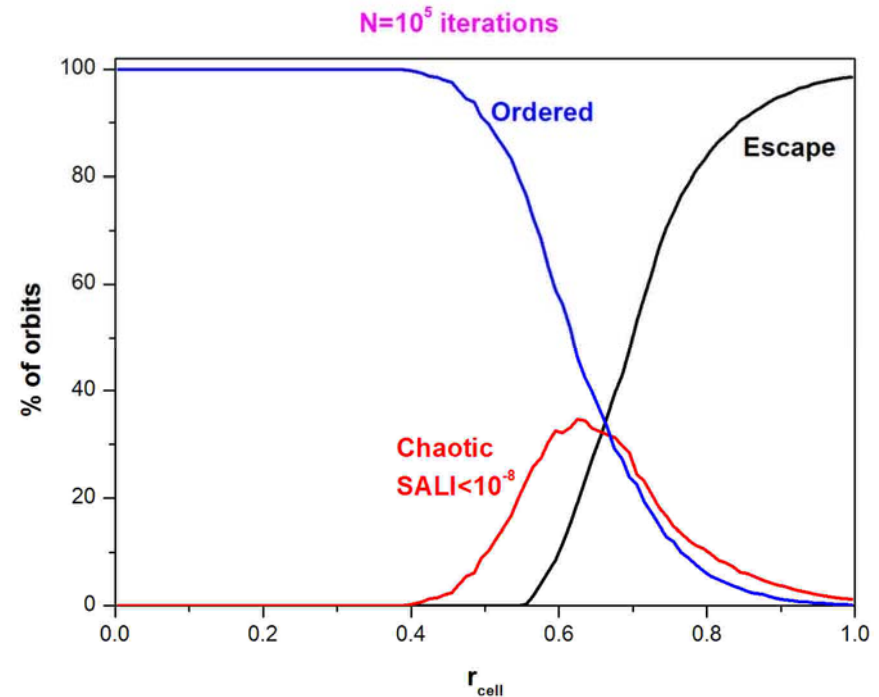
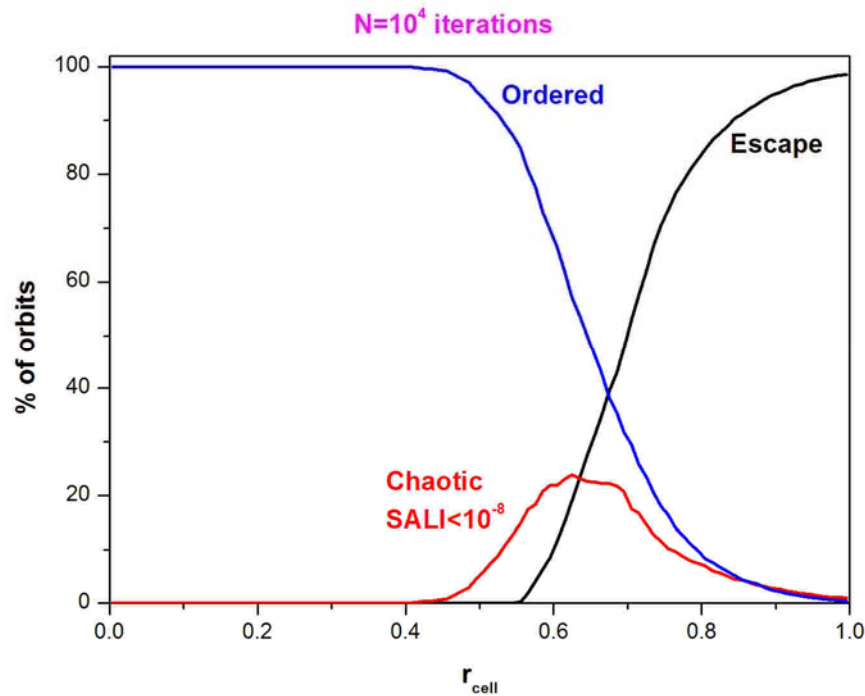
4D Accelerator map – "Global" study

Regions of **different values of the SALI** on the subspace $x_2(0)=x_4(0)=0$, after 10^4 iterations ($q_x=0.61803$ $q_y=0.4152$)



4D Accelerator map – "Global" study

We consider 1,922,833 orbits by varying all x_1, x_2, x_3, x_4 within spherical shells of width 0.01 in a hypersphere of radius 1. ($q_x=0.61803$ $q_y=0.4152$)



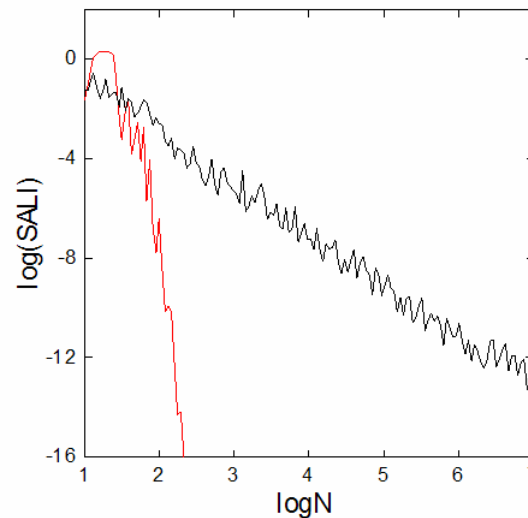
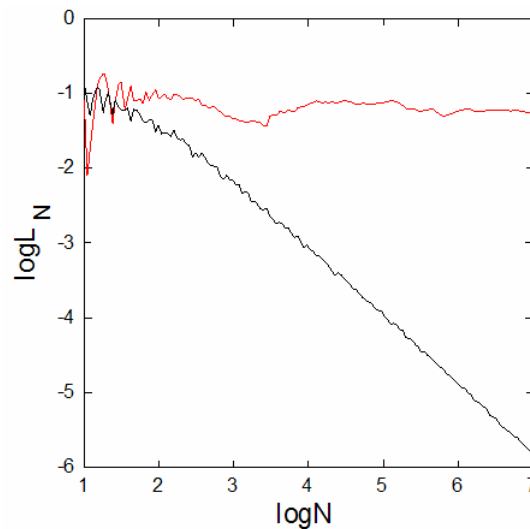
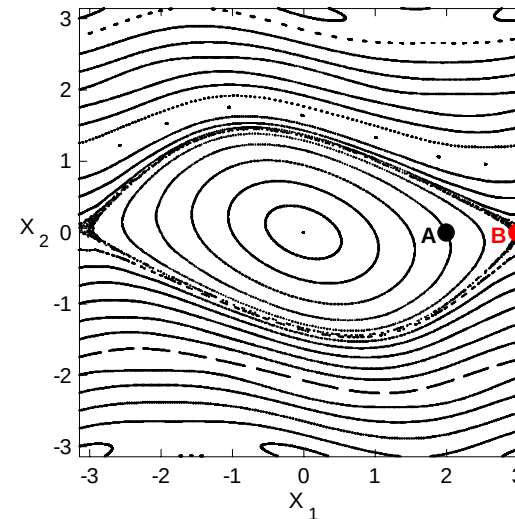
Applications – 2D map

$$\begin{aligned}x'_1 &= x_1 + x_2 \\x'_2 &= x_2 - v \sin(x_1 + x_2)\end{aligned}\quad (\text{mod } 2\pi)$$

For $v=0.5$ we consider the orbits:

ordered orbit A with initial conditions $x_1=2, x_2=0$.

chaotic orbit B with initial conditions $x_1=3, x_2=0$.



Behavior of SALI

2D maps

SALI $\rightarrow 0$ both for regular and chaotic orbits

following, however, completely different time rates which allows us to distinguish between the two cases.

Hamiltonian flows and multidimensional maps

SALI $\rightarrow 0$ for chaotic orbits

SALI $\rightarrow \text{constant} \neq 0$ for regular orbits

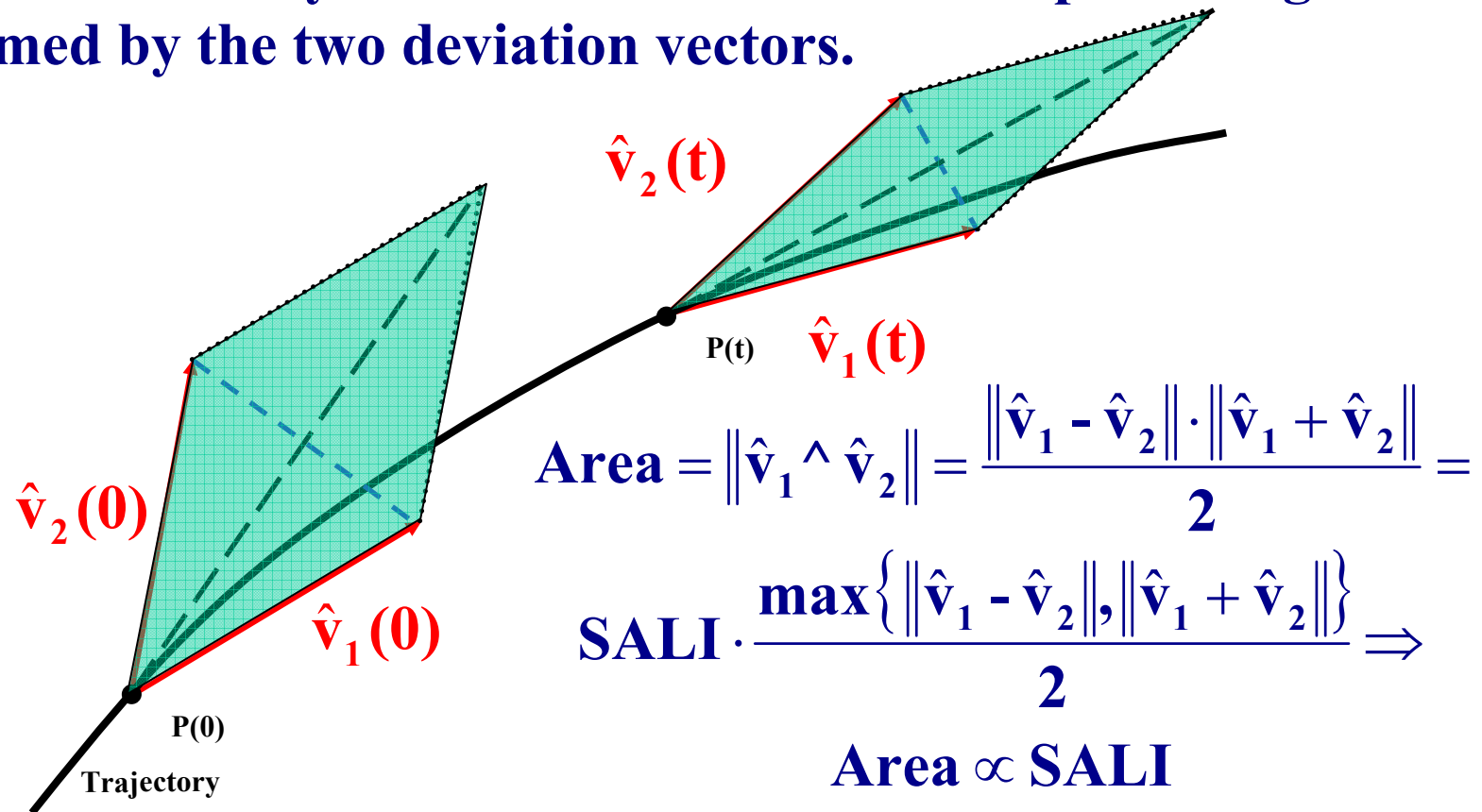
Questions

Can we generalize SALI so that the new index:

- Can rapidly reveal the nature of chaotic orbits with $\sigma_1 \approx \sigma_2$ ($\text{SALI} \propto e^{-(\sigma_1 - \sigma_2)t}$)?
- Depends on several Lyapunov exponents for chaotic orbits?
- Exhibits power-law decay for regular orbits depending on the dimensionality of the tangent space of the reference orbit as for 2D maps?

Definition of Generalized Alignment Index (GALI)

SALI effectively measures the ‘area’ of the parallelogram formed by the two deviation vectors.



Definition of GALI

In the case of an N degree of freedom Hamiltonian system or a $2N$ symplectic map we follow the evolution of

k deviation vectors with $2 \leq k \leq 2N$,

and define (Skokos et al., 2007, Physica D, 231, 30) the Generalized Alignment Index (GALI) of order k :

$$\text{GALI}_k(t) = \|\hat{\mathbf{v}}_1(t) \wedge \hat{\mathbf{v}}_2(t) \wedge \dots \wedge \hat{\mathbf{v}}_k(t)\|$$

where

$$\hat{\mathbf{v}}_1(t) = \frac{\mathbf{v}_1(t)}{\|\mathbf{v}_1(t)\|}$$

Wedge product

We consider as a basis of the $2N$ -dimensional tangent space of the Hamiltonian flow the usual set of orthonormal vectors:

$$\hat{e}_1 = (1, 0, 0, \dots, 0), \hat{e}_2 = (0, 1, 0, \dots, 0), \dots, \hat{e}_{2N} = (0, 0, 0, \dots, 1)$$

Then for k deviation vectors we have:

$$\begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \vdots \\ \hat{v}_k \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1,2N} \\ v_{21} & v_{22} & \cdots & v_{2,2N} \\ \vdots & \vdots & & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{k,2N} \end{bmatrix} \cdot \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_{2N} \end{bmatrix}$$

$$\hat{v}_1 \wedge \hat{v}_2 \wedge \cdots \wedge \hat{v}_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2N} \begin{vmatrix} v_{1i_1} & v_{1i_2} & \cdots & v_{1i_k} \\ v_{2i_1} & v_{2i_2} & \cdots & v_{2i_k} \\ \vdots & \vdots & & \vdots \\ v_{ki_1} & v_{ki_2} & \cdots & v_{ki_k} \end{vmatrix} \hat{e}_{i_1} \wedge \hat{e}_{i_2} \wedge \cdots \wedge \hat{e}_{i_k}$$

Computation of GALI

For k deviation vectors:

$$\begin{bmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \\ \vdots \\ \hat{\mathbf{v}}_k \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} & \cdots & \mathbf{v}_{12N} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \cdots & \mathbf{v}_{22N} \\ \vdots & \vdots & & \vdots \\ \mathbf{v}_{k1} & \mathbf{v}_{k2} & \cdots & \mathbf{v}_{k2N} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \vdots \\ \hat{\mathbf{e}}_{2N} \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \vdots \\ \hat{\mathbf{e}}_{2N} \end{bmatrix}$$

the ‘norm’ of the wedge product is given by:

$$\|\hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2 \wedge \cdots \wedge \hat{\mathbf{v}}_k\| = \left\{ \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2N} \left| \begin{bmatrix} \mathbf{v}_{1i_1} & \mathbf{v}_{1i_2} & \cdots & \mathbf{v}_{1i_k} \\ \mathbf{v}_{2i_1} & \mathbf{v}_{2i_2} & \cdots & \mathbf{v}_{2i_k} \\ \vdots & \vdots & & \vdots \\ \mathbf{v}_{ki_1} & \mathbf{v}_{ki_2} & \cdots & \mathbf{v}_{ki_k} \end{bmatrix} \right|^2 \right\}^{1/2} = \sqrt{\det(\mathbf{A} \cdot \mathbf{A}^T)}$$

Computation of GALI

From **Singular Value Decomposition (SVD)** of A^T we get:

$$A^T = U \cdot W \cdot V^T$$

where U is a column-orthogonal $2N \times k$ matrix ($U^T \cdot U = I$), V^T is a $k \times k$ orthogonal matrix ($V \cdot V^T = I$), and W is a diagonal $k \times k$ matrix with positive or zero elements, the so-called **singular values**. So, we get:

$$\begin{aligned} \det(A \cdot A^T) &= \det(V \cdot W^T \cdot U^T \cdot U \cdot W \cdot V^T) = \det(V \cdot W \cdot I \cdot W \cdot V^T) = \\ \det(V \cdot W^2 \cdot V^T) &= \det(V \cdot \text{diag}(w_1^2, w_2^2, \dots, w_k^2) \cdot V^T) = \prod_{i=1}^k w_i^2 \end{aligned}$$

Thus, GALI_k is computed by:

$$\text{GALI}_k = \sqrt{\det(A \cdot A^T)} = \prod_{i=1}^k w_i \Rightarrow \log(\text{GALI}_k) = \sum_{i=1}^k \log(w_i)$$

Behavior of $GALI_k$ for chaotic motion

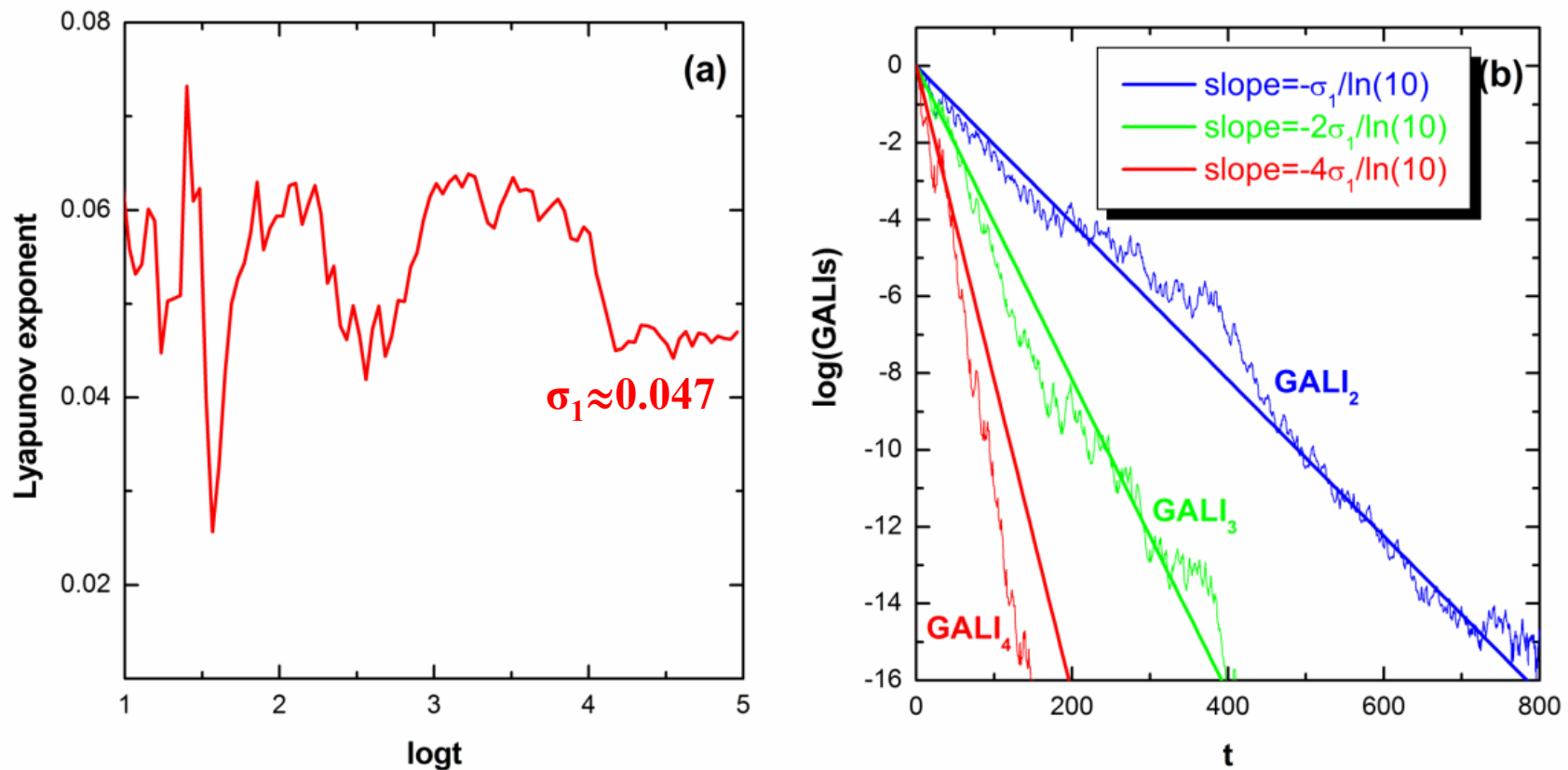
$GALI_k$ ($2 \leq k \leq 2N$) tends exponentially to zero with exponents that involve the values of the first k largest Lyapunov exponents $\sigma_1, \sigma_2, \dots, \sigma_k$:

$$GALI_k(t) \propto e^{-[(\sigma_1 - \sigma_2) + (\sigma_1 - \sigma_3) + \dots + (\sigma_1 - \sigma_k)]t}$$

The above relation is valid even if some Lyapunov exponents are equal, or very close to each other.

Behavior of $GALI_k$ for chaotic motion

2D Hamiltonian (Hénon-Heiles system)

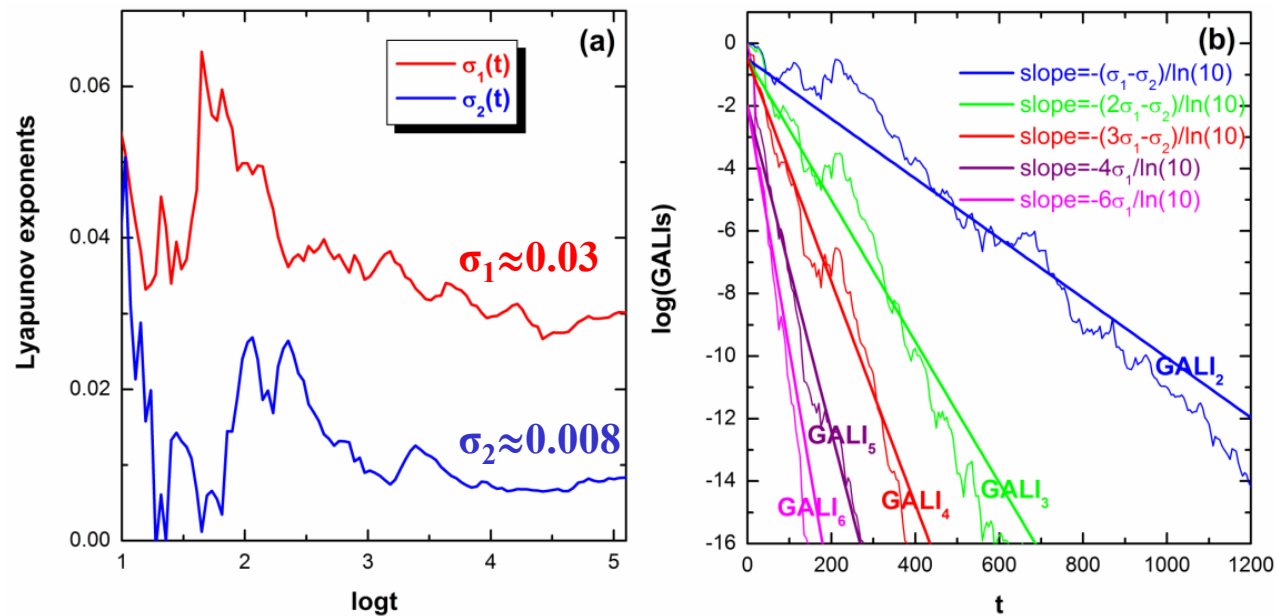


Behavior of $GALI_k$ for chaotic motion

3D system:

$$H_3 = \sum_{i=1}^3 \frac{\omega_i}{2} (q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

with $\omega_1=1$, $\omega_2=\sqrt{2}$, $\omega_3=\sqrt{3}$, $H_3=0.09$.

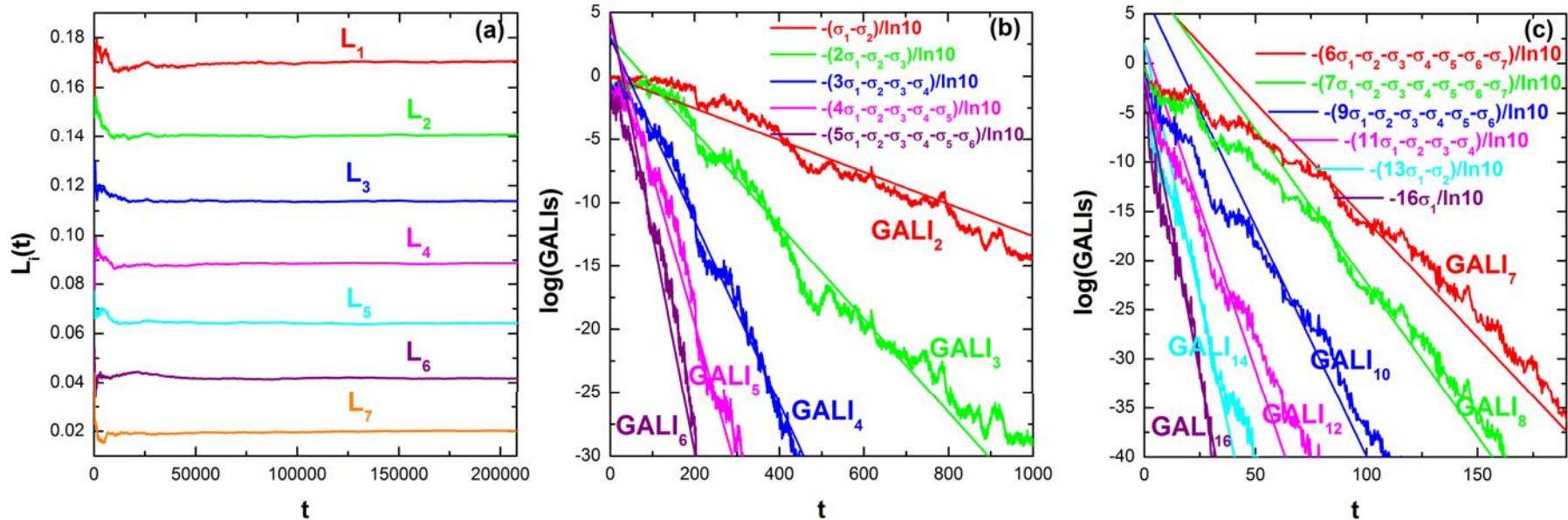


Behavior of GALI_k for chaotic motion

N particles Fermi-Pasta-Ulam (FPU) system:

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 + \sum_{i=0}^N \left[\frac{1}{2} (q_{i+1} - q_i)^2 + \frac{\beta}{4} (q_{i+1} - q_i)^4 \right]$$

with fixed boundary conditions, $N=8$ and $\beta=1.5$.



Behavior of GALI_k for regular motion

If the motion occurs on an s -dimensional torus with $s \leq N$ then the behavior of GALI_k is given by (Skokos et al., 2008, EPJ-ST, 165, 5):

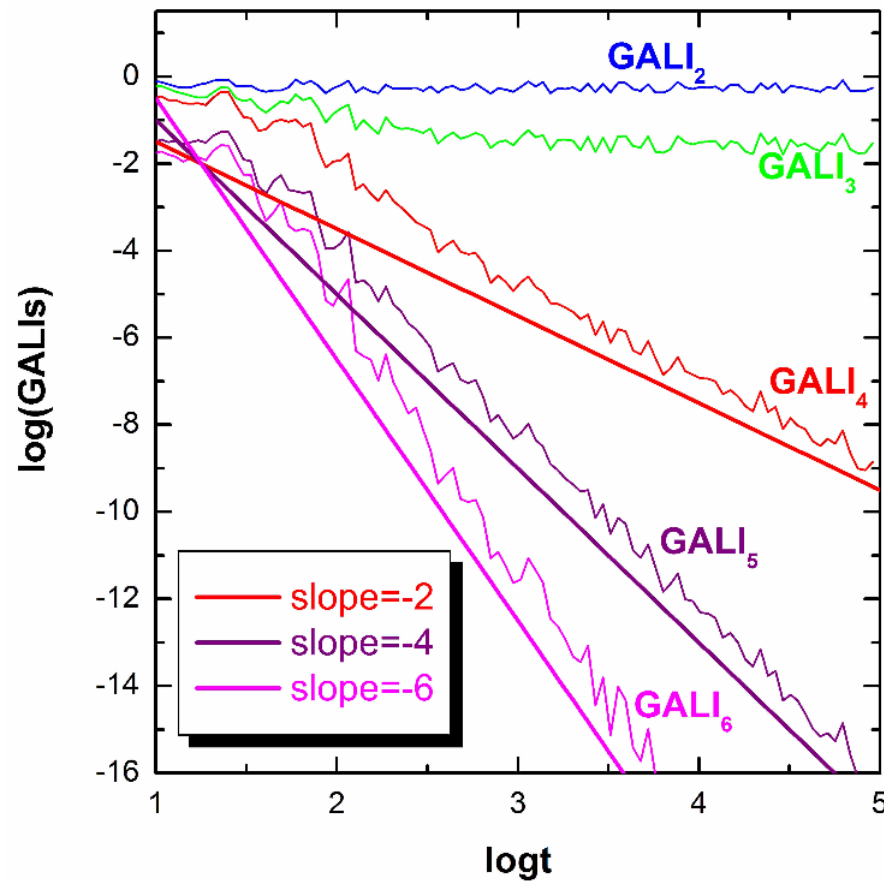
$$\text{GALI}_k(t) \propto \begin{cases} \text{constant} & \text{if } 2 \leq k \leq s \\ \frac{1}{t^{k-s}} & \text{if } s < k \leq 2N - s \\ \frac{1}{t^{2(k-N)}} & \text{if } 2N - s < k \leq 2N \end{cases}$$

while in the common case with $s=N$ we have :

$$\text{GALI}_k(t) \propto \begin{cases} \text{constant} & \text{if } 2 \leq k \leq N \\ \frac{1}{t^{2(k-N)}} & \text{if } N < k \leq 2N \end{cases}$$

Behavior of GALI_k for regular motion

3D Hamiltonian



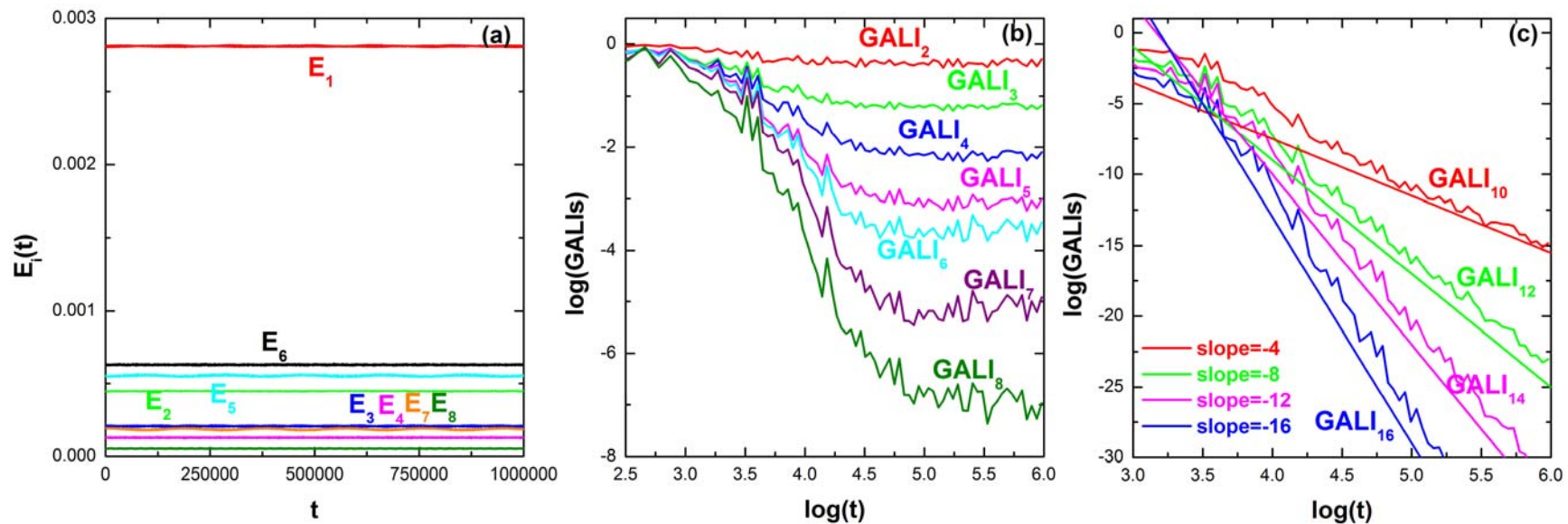
Behavior of GALI_k for regular motion

N=8 FPU system: The unperturbed Hamiltonian ($\beta=0$) is written as a sum of the so-called **harmonic energies** E_i :

$$E_i = \frac{1}{2} (P_i^2 + \omega_i^2 Q_i^2), \quad i = 1, \dots, N$$

with:

$$Q_i = \sqrt{\frac{2}{N+1}} \sum_{i=1}^N q_i \sin\left(\frac{ki\pi}{N+1}\right), \quad P_i = \sqrt{\frac{2}{N+1}} \sum_{i=1}^N p_i \sin\left(\frac{ki\pi}{N+1}\right), \quad \omega_i = 2 \sin\left(\frac{i\pi}{2(N+1)}\right)$$



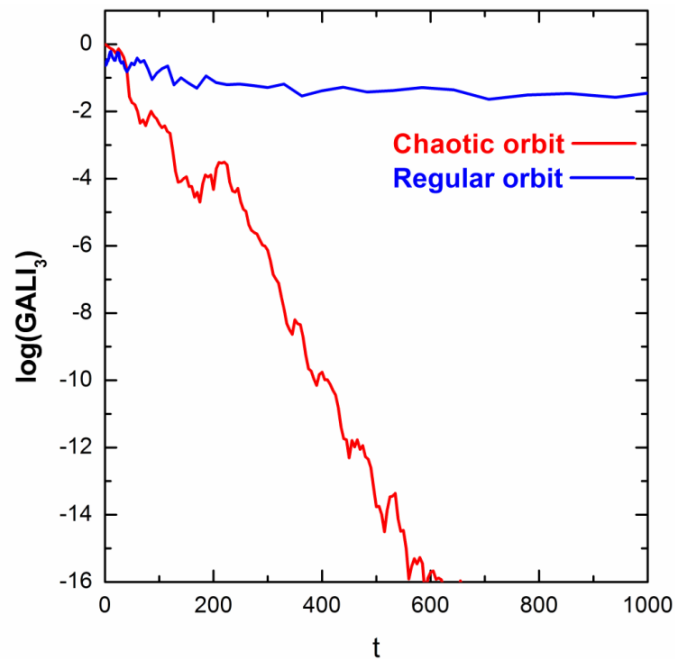
Global dynamics

- GALI_2 (practically equivalent to the use of SALI)

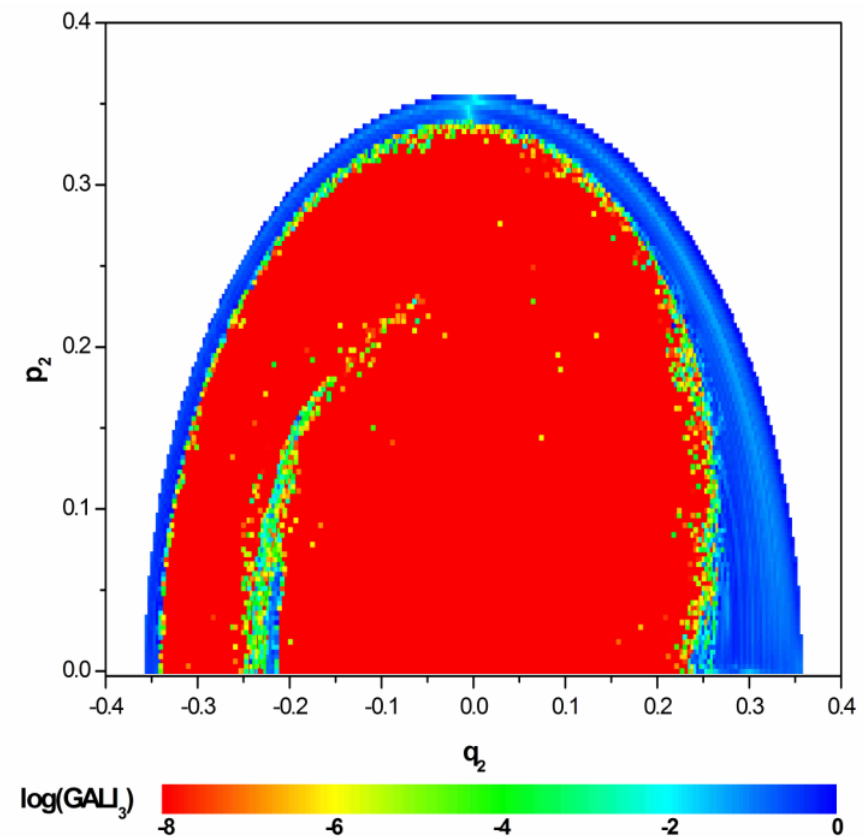
- GALI_N

Chaotic motion: $\text{GALI}_N \rightarrow 0$
(exponential decay)

Regular motion:
 $\text{GALI}_N \rightarrow \text{constant} \neq 0$



3D Hamiltonian
Subspace $q_3=p_3=0$, $p_2 \geq 0$ for $t=1000$.



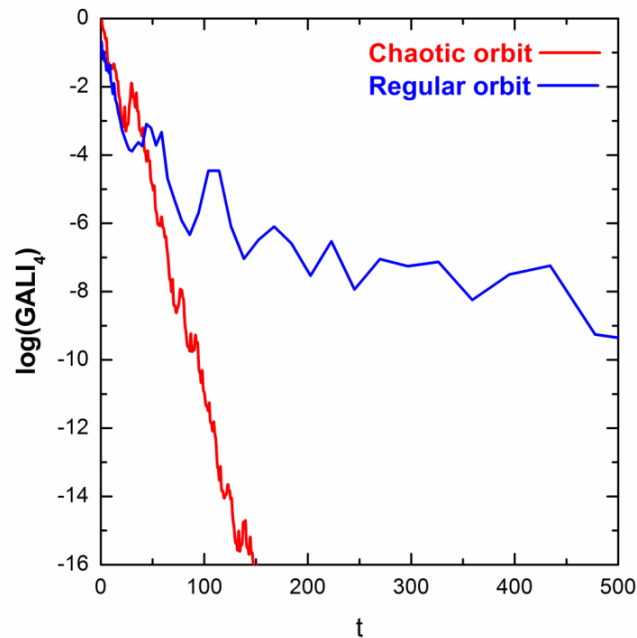
Global dynamics

$GALI_k$ with $k > N$

The index tends to zero both for regular and chaotic orbits but with completely different time rates:

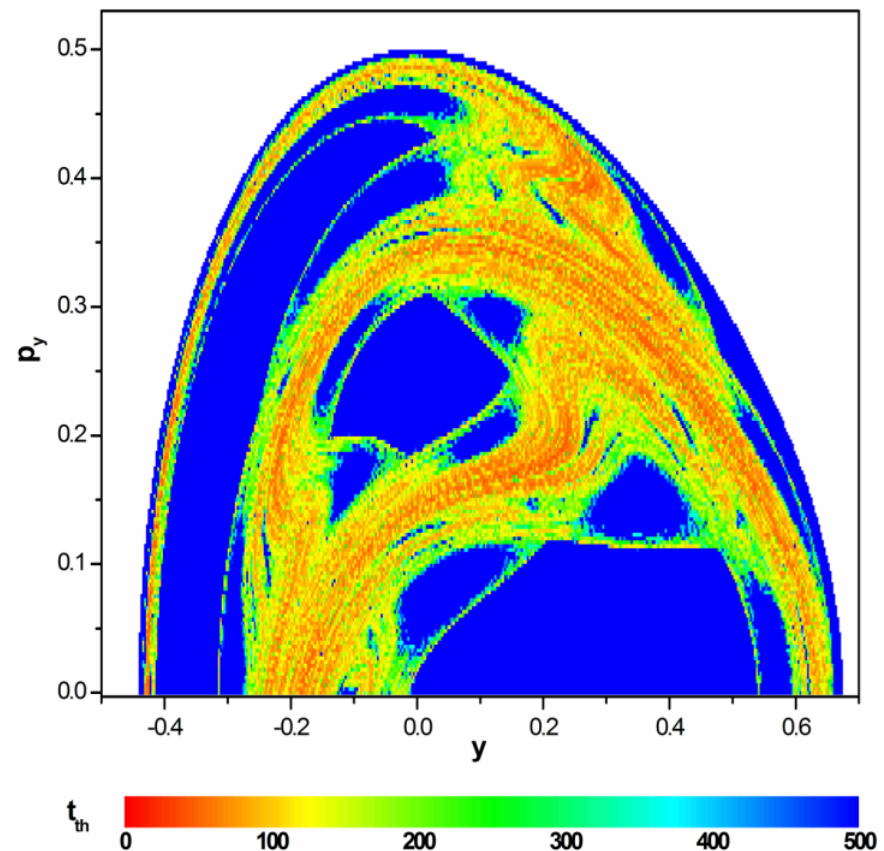
Chaotic motion: exponential decay

Regular motion: power law



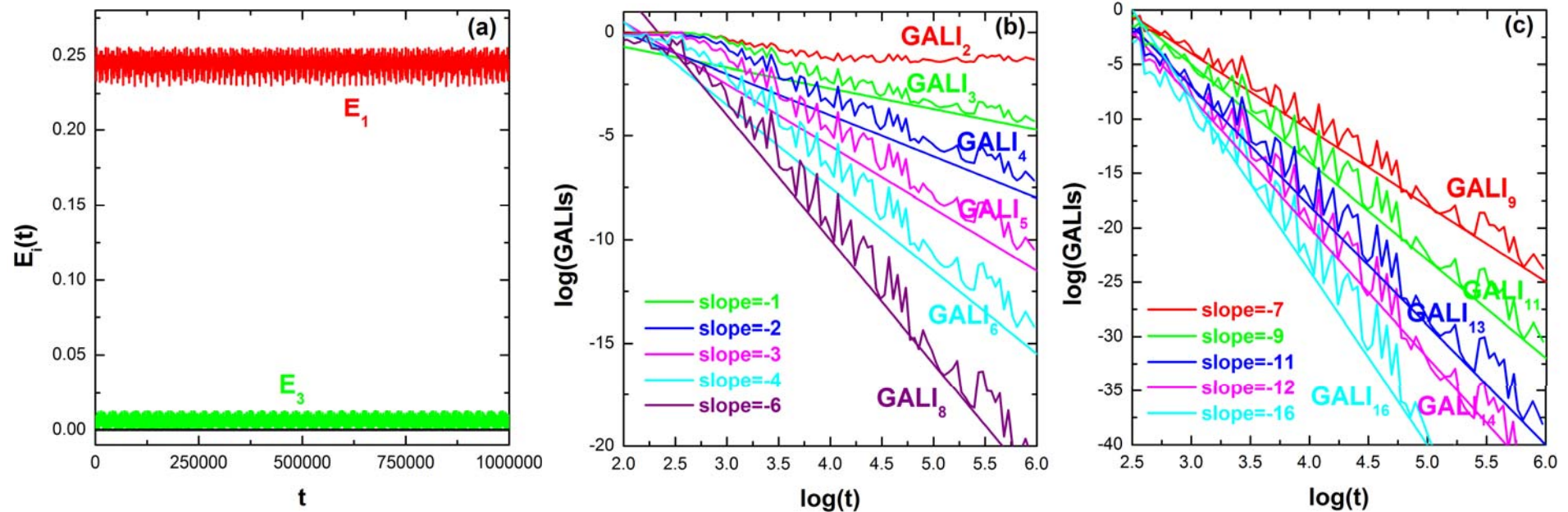
2D Hamiltonian (Hénon-Heiles)

Time needed for $GALI_4 < 10^{-12}$



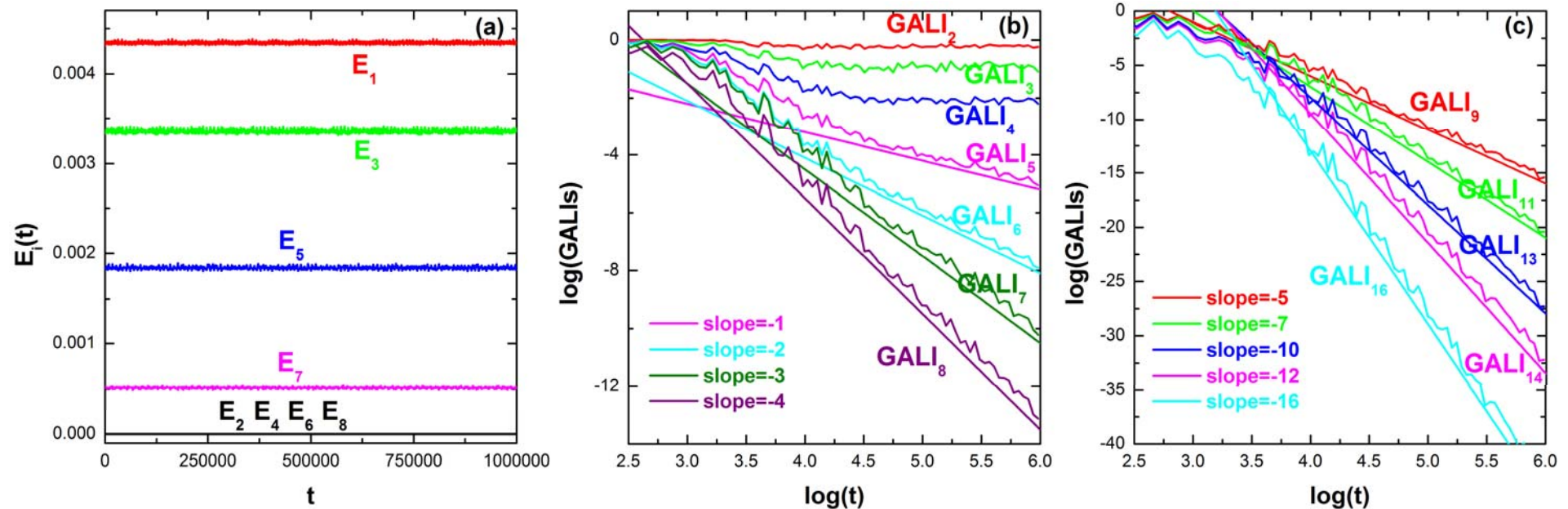
Regular motion on low-dimensional tori

A regular orbit lying on a **2-dimensional torus** for the N=8 FPU system.



Regular motion on low-dimensional tori

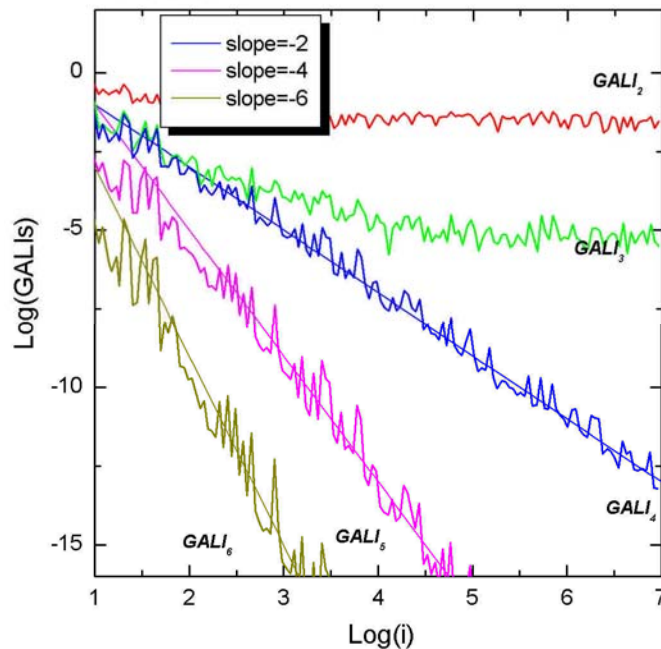
A regular orbit lying on a **4-dimensional torus** for the $N=8$ FPU system.



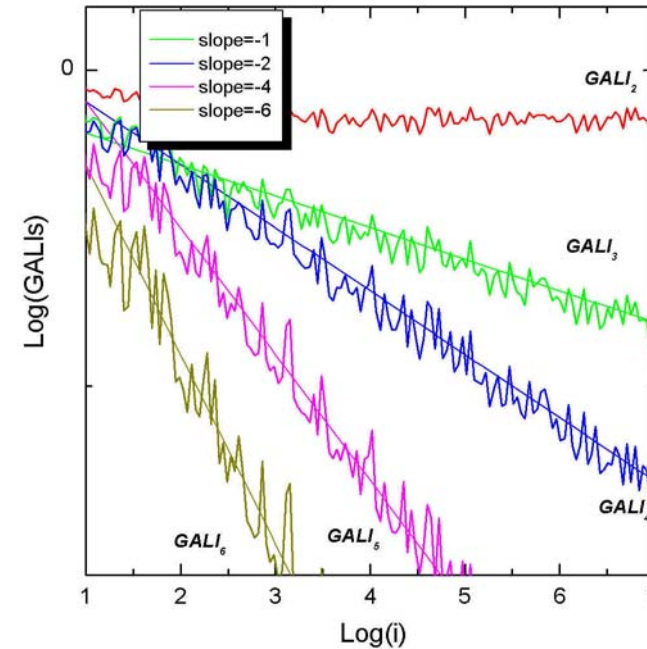
Low-dimensional tori - 6D map

$$\begin{aligned}
 x'_1 &= x_1 + x'_2 \\
 x'_2 &= x_2 + \frac{K_1}{2\pi} \sin(2\pi x_1) - \frac{B}{2\pi} \{ \sin[2\pi(x_5 - x_1)] + \sin[2\pi(x_3 - x_1)] \} \\
 x'_3 &= x_3 + x'_4 \\
 x'_4 &= x_4 + \frac{K_2}{2\pi} \sin(2\pi x_3) - \frac{B}{2\pi} \{ \sin[2\pi(x_1 - x_3)] + \sin[2\pi(x_5 - x_3)] \} \pmod{1} \\
 x'_5 &= x_5 + x'_6 \\
 x'_6 &= x_6 + \frac{K_3}{2\pi} \sin(2\pi x_5) - \frac{B}{2\pi} \{ \sin[2\pi(x_3 - x_5)] + \sin[2\pi(x_1 - x_5)] \}
 \end{aligned}$$

3D torus



2D torus



Behavior of $GALI_k$

Chaotic motion:

$GALI_k \rightarrow 0$ exponential decay

$$GALI_k(t) \propto e^{-[(\sigma_1 - \sigma_2) + (\sigma_1 - \sigma_3) + \dots + (\sigma_1 - \sigma_k)]t}$$

Regular motion:

$GALI_k \rightarrow \text{constant} \neq 0$ or $GALI_k \rightarrow 0$ power law decay

$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if } 2 \leq k \leq s \\ \frac{1}{t^{k-s}} & \text{if } s < k \leq 2N-s \\ \frac{1}{t^{2(k-N)}} & \text{if } 2N-s < k \leq 2N \end{cases}$$

Conclusions

- Generalizing the SALI method we define the Generalized ALignment Index of order k ($GALI_k$) as the volume of the generalized parallelepiped, whose edges are k unit deviation vectors. $GALI_k$ is computed as the product of the singular values of a matrix (SVD algorithm).
- Behaviour of $GALI_k$:
 - ✓ **Chaotic motion**: it tends exponentially to zero with exponents that involve the values of several Lyapunov exponents.
 - ✓ **Reguler motion**: it fluctuates around non-zero values for $2 \leq k \leq s$ and goes to zero for $s < k \leq 2N$ following power-laws, with s being the dimensionality of the torus.
- $GALI_k$ indices :
 - ✓ can distinguish rapidly and with certainty between regular and chaotic motion
 - ✓ can be used to characterize individual orbits as well as "chart" chaotic and regular domains in phase space.
 - ✓ are perfectly suited for studying the global dynamics of multidimentonal systems
 - ✓ can identify regular motion in low-dimensional tori

References

- **SALI**
 - ✓ Skokos Ch. (2001) J. Phys. A, 34, 10029
 - ✓ Skokos Ch., Antonopoulos Ch., Bountis T. C. & Vrahatis M. N. (2003) Prog. Theor. Phys. Supp., 150, 439
 - ✓ Skokos Ch., Antonopoulos Ch., Bountis T. C. & Vrahatis M. N. (2004) J. Phys. A, 37, 6269
- **GALI**
 - ✓ Skokos Ch., Bountis T. C. & Antonopoulos Ch. (2007) Physica D, 231, 30-54
 - ✓ Skokos Ch., Bountis T. C. & Antonopoulos Ch. (2008) Eur. Phys. J. Sp. Top., 165, 5-14